Spherically Symmetric Solution for Torsion and the Dirac Equation in 5D Spacetime

Dzhunushaliev V.D. *

Deprt. of Phys., VCU, Richmond, VA 23284-2000, USA

and Theor. Phys. Dept., KSNU, Bishkek, 720024, Kyrgyzstan

Abstract

Torsion in a 5D spacetime is considered. In this case gravitation is defined by the 5D metric and the torsion. It is conjectured that torsion is connected with a spinor field. In this case Dirac's equation becomes the nonlinear Heisenberg equation. It is shown that this equation has a discrete spectrum of solutions with each solution being regular on the whole space and having finite energy. Every solution is concentrated on the Planck region and hence we can say that torsion should play an important role in quantum gravity in the formation of bubbles of spacetime foam. On the basis of the algebraic relation between torsion and the classical spinor field in Einstein-Cartan gravity the geometrical interpretation of the spinor field is considered as "the square root" of torsion.

*E-mail address: dzhun@freenet.bishkek.su

I. INTRODUCTION

It is well known [1], [2], [3] that the inclusion of torsion leads to the result that in the Dirac equation there appears a nonlinear term. Heisenberg was the first researcher to investigate such a nonlinear equation [4], [5]. He assumed that a quantization of this nonlinear equation would give the mass spectrum of elementary particles. This attempt was not successful. At present it is understood that the Heisenberg equation with the specific nonlinear term, $(\overline{\psi}\gamma^\mu\gamma^5\psi)\gamma_\mu\gamma^5\psi$, is the consequence of including torsion in the geometry of spacetime. On the other hand it is well known that the 4D Dirac equation can be naturally generalized to a 5D equation. In this paper the Dirac equation is considered in 5D spacetime with torsion. This 5D Heisenberg equation has a discrete spectrum of regular solutions with finite energy. Since the kernel of this solution is concentrated in a region whose size is comparable with the Planck length we can postulate that torsion plays important role in the formation of spacetime foam in quantum gravity.

II. 5D HEISENBERG EQUATION

In this paper the 5D spacetime is the principal bundle with the $U(1)$ structural group and Einstein's 4D spacetime as the base of this bundle [6]. This means that 5D spacetime M^5 (total space of the bundle) can be presented locally as $M^5 = M^4 \times U(1)$ where M^4 is the Einstein's spacetime (base of the bundle) and $U(1)$ is the structural group of the bundle (fibre of the bundle). The total space M^5 is symmetric under the action of the structural group $U(1)$ (gauge group) and $M^4 = M^5/U(1)$. In this case the extra dimension is the symmetrical space (gauge group) and the 5D metric on the total space has the following

form:

$$ds^2 = e^{2\phi(x^\alpha)}\left(dx^5 - A_\mu(x^\alpha)dx^\mu\right)^2 + g_{\mu\nu}(x^\alpha)dx^\mu dx^\nu, \tag{1}$$

here $g_{\mu\nu}$ is the 4D metric on the base of the bundle; $\alpha, \mu, \nu = 0, 1, 2, 3$ are the spacetime indices; A_μ is the electromagnetic potential according to the following theorem [7], [8]:

Let G be the group fibre of the principal bundle. Then there is a one-to-one correspondence between the G-invariant metrics on the total space $\mathcal{X}$ and the triples $(g_{\mu\nu}, A^a_\mu, h\gamma_{ab})$. Here $g_{\mu\nu}$ is Einstein's pseudo - Riemannian metric on the base; A^a_μ is the gauge field of the group G (the nondiagonal components of the multidimensional metric); $h\gamma_{ab}$ is the symmetric metric on the fibre.

In this case any physical fields will not depend on the 5^{th} coordinate introduced on the 5^{th} dimension. The Lagrangian for the Dirac spinor field in this spacetime with gravity and torsion can be written in the following manner:

$$L = \frac{\hbar c}{2}\left[i\overline{\psi}\gamma^A(\nabla_A\psi) + \frac{mc}{\hbar}\overline{\psi}\psi + (Hermitian - conjugate)\right] + \frac{1}{2k}R, \tag{2}$$

here $A, B, C = 0, 1, 2, 3, 4$ are 5D spacetime indices on the total space of bundle; γ^A are Dirac matrics satisfying the following condition $\{\gamma^A, \gamma^B\} = 2G^{AB}$; G^{AB} is the 5D spacetime metric; $k = 8\pi G/c^4$; R is the 5D Ricci scalar of the affine connection $\Gamma^A_{\bullet BC}$ (all definitions for Riemann-Cartan geometry given in Appendix A). The covariant derivative of the spinor field is defined in the following way [2], [9]:

$$\nabla_A\psi = \left(\partial_A - \frac{1}{4}\omega_{abA}\gamma^{[a}\gamma^{b]} - \frac{1}{4}S_{abA}\gamma^{[a}\gamma^{b]}\right)\psi, \tag{3}$$

here $a, b = 0, 1, 2, 3, 4$ are five-bein indices; $\gamma^a = \gamma^0, \gamma^1, \gamma^2, \gamma^3, \gamma^4 = \gamma^5$ are ordinary Dirac matrices $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$; $\eta^{ab} = diag\{1, -1, -1, -1, -1\}$ is the 5D Minkowski metric; [ ]

means antisymmetrization; { } is symmetrization. The coefficients of the spinor connection are defined as follows:

$$\omega_{abA} = h_{aB} h_b^{\bullet C} \left\{ {B \atop AC} \right\} + h_a^{\bullet B} \frac{\partial h_{bB}}{\partial x^A} \tag{4}$$

here $h^a_{\bullet A}$ is a five-bein, $\left\{ {B \atop AC} \right\}$ Christoffel symbols. We note that for the spinor field defined on the total space of principal bundle the covariant derivative with respect to extra coordinate (along to fibre of principal bundle) is:

$$\nabla_5 \psi = -\frac{1}{4} \left(\omega_{abA} \gamma^{[a} \gamma^{b]} + S_{abA} \gamma^{[a} \gamma^{b]} \right) \psi, \tag{5}$$

because $\partial_5 \psi = 0$ as was indicated above. For the totally antisymmetric torsion varying the torsion, spinor fields and metric leads to the following fields equations:

$$S^{abc} = -4 i l_{Pl}^2 \left(\overline{\psi} \gamma^{[a} \gamma^b \gamma^{c]} \psi \right), \tag{6}$$

$$\left(i \gamma^A \partial_A - \frac{i}{4} \omega_{abA} \gamma^A \gamma^{[a} \gamma^{b]} - l_{Pl}^2 (\overline{\psi} \gamma^{[a} \gamma^b \gamma^{c]} \psi) \gamma_{[a} \gamma_b \gamma_{c]} + \frac{mc}{\hbar} \right) \psi = 0, \tag{7}$$

$$R_{AB} - \frac{1}{2} G_{AB} R = 8 l_{Pl}^2 T^D_{AB} + 8 l_{Pl}^4 G_{AB} (\overline{\psi} \gamma^{[a} \gamma^b \gamma^{c]} \psi)(\overline{\psi} \gamma_{[a} \gamma_b \gamma_{c]} \psi), \tag{8}$$

here S^{abc} is the antisymmetric torsion tensor, $l_{Pl}^2 = \pi \hbar G / c^3$ (l_{Pl} is Planck length), R_{AB} is the 5D Ricci tensor. The stress-energy tensor T^D_{AB} of the Dirac field is:

$$T^D_{AB} = -i \left[\overline{\psi} \gamma_A (\overset{\{\}}{\nabla}_B \psi) + \overline{\psi} \gamma_B (\overset{\{\}}{\nabla}_A \psi) \right] + (Hermitian - conjugate), \tag{9}$$

here $\overset{\{\}}{\nabla}_A$ means the covariant derivative without torsion. The 5D metric is defined in the ordinary way $G_{AB} = h^a_{\bullet B} h_{aB}$. We suppose that in this spacetime there are small regions where torsion plays the key role. In this case equation (7) is the key equation for understanding what happens in such small regions. In this paper we want to demonstrate that the 5D Heisenberg equation, taking into account torsion, has solutions which are regular in the whole spacetime where the gravitational effect comes only from torsion.

The ansatz for equation (7) is taken as the standard spherically symmetric spinor:

$$\psi(r,t) = e^{i\omega t} \begin{Bmatrix} f(r) \\ 0 \\ ig(r)\cos\theta \\ ig(r)\sin\theta e^{i\varphi} \end{Bmatrix}, \tag{10}$$

here r, θ, φ are the spherical coordinates. It is interesting to note that in the 4D case the scalar $(\overline{\psi}\gamma^{[a}\gamma^b\gamma^{c]}\psi)(\overline{\psi}\gamma_{[a}\gamma_b\gamma_{c]}\psi)$ is not spherically symmetric. The nonlinear term $(\overline{\psi}\gamma^{[a}\gamma^b\gamma^{c]}\psi)(\gamma_{[a}\gamma_b\gamma_{c]}\psi)$ in the Heisenberg equation is spherical symmetric only in 5D spacetime (see Appendix B). This is highly unusual and evidently points ***to the close connection between torsion and multidimensional gravity.*** The substitution of Eq. (10) into Heisenberg's Eq. (7) gives us the following two equations (remember that we ignore the gravitational effects connected with the metric):

$$g' + (-m+\omega)f + \frac{2g}{r} - 12 l_{Pl}^2 f\left(f^2 - g^2\right) = 0, \tag{11}$$

$$f' - (m+\omega)g - 12 l_{Pl}^2 g\left(f^2 - g^2\right) = 0 \tag{12}$$

here $\hbar, c = 1$. These equations coincide identically with equations for the 4D Heisenberg equation which have been investigated in Refs [10]- [11] with other nonlinear terms ($|\overline{\psi}\psi|^2$ and $|\overline{\psi}\gamma^\mu\psi|^2$). Equations (11) - (12) have regular solutions in all space only for some discrete set of initial values $f(0)$ and $g(0)$. Near the origin the regular solution has the following behavior:

$$g(r) = g_1 r + g_3 \frac{r^3}{6} + \cdots, \tag{13}$$

$$f(r) = f_0 + f_2 \frac{r^2}{2} + \cdots \tag{14}$$

The substitution (13) -(14) into Eqs (11) - (12) give us:

$$g_1 = \frac{f_0}{3}\left[12 l_{Pl}^2 f_0^2 + (m-\omega)\right], \tag{15}$$

$$f_2 = g_1\left[12 l_{Pl}^2 f_0^2 + (m+\omega)\right]. \tag{16}$$

This means that for fixed m and ω a solution depends only on the initial value $f(0) = f_0$. For arbitrary f_0 values the solution is singular at infinity ($r \to \infty$). However there are a discrete series of $(f_0)_n$ values for which the solutions become regular at infinity (n is the number of intersections that $f(r)$ ($g(r)$) make with the r-axe). Each of these solutions has finite mass and spin. The ground state is for $n = 0$. More detailed discussion about properties of these solutions can be found in Ref. [10], [11]. Thus, Eqs (11) and (12) have a discrete spectrum of regular solutions in all space, and they have finite energy. At infinity ($r \to \infty$) these solutions have the following asymptotical behaviour:

$$f = f_\infty + \frac{a e^{-\alpha r}}{r^2} + \cdots, \tag{17}$$

$$g = \frac{b e^{-\alpha r}}{r^2} + \cdots, \tag{18}$$

$$\alpha^2 = 4\omega(m + \omega), \tag{19}$$

$$\frac{b}{a} = -\sqrt{1 - \frac{m}{\omega}}, \tag{20}$$

$$f_\infty = \pm\sqrt{\frac{-m + \omega}{12 l_{Pl}^2}}. \tag{21}$$

This guarantees the finiteness of mass, energy and so on. If the energy of such solutions is $\omega \approx E_{Pl} = (\hbar c^5/G)^{1/2}$ and $m = 0$ then these solutions are essentially nonzero only in the Planck region.

III. DISCUSSION

1. We see that the linear size of these solutions is on the order of the Planck length. The energy, spin and action are finite for these solutions and they are concentrated in the Planck region. This means that these solutions will give an essential contribution to the Feynman path integral in quantum gravity. All these results allow us to hypothesize that torsion can play a very important role in the formation of bubbles in the spacetime foam and in preventing cosmological singularities [12], [13]. In conclusion I would like to underline that probably ***torsion + multidimensional gravity*** is a more natural object in nature than either object separately.

2. It is well known that boson (gauge) fields have a geometrical interpretation in multidimensional Kaluza-Klein theories as off-diagonal components of a multidimensional metric. Unfortunately the fermion fields do not have a similar geometrical interpretation. This is a big obstacle to Einstein's point of view that nature is ***pure geometry***. It is possible that Eq. (6) can be considered as an algebraic relation for geometrization of the classical spinor field. Let us examine it not in terms of the right side (spinor field) as the source for the left side (torsion) but *vice versa*: torsion S^{abc} is the source of the classical spinor field ψ. In this case we can say that the classical spinor field in some sense is "the square root" of torsion just the Dirac equation is "the square root" of the Klein-Gordon equation. Such a point of view immediately leads to the conclusion that ***nonpropagating torsion in Einstein-Cartan gravity is the geometrical source for fermion fields.*** In 4D gravity Eq. (6) can be written as $S^\mu \propto \overline{\psi}\gamma^\mu\gamma^5\psi$ where in the left and right sides of the equation we have 4 number of independent components. In order to define the 4 spinor components we have 4 quadratic algebraic equations. In this case the Heisenberg equation (7) is the gravity equation! In the 5D case we can introduce the tensor Σ^{ab} instead of S^{abc}: $\Sigma^{ab} = E^{abcde}S_{cde}$ (here E^{abcde} is the 5D completely antisymmetrical Levi-Civita tensor). The antisymmetrical tensor Σ^{ab} has the $\frac{(5\times5)-5}{2} = 10$ number of independent components. Hence each component of the spinor would be some nonlinear combination of 10 torsion components. But in fact the Heisenberg equation (7) has only 4 components, hence for geometrization of the spinor field in 5D space we should have torsion with some algebraic restrictions on torsion.

IV. ACKNOWLEDGMENTS

This project has been funded by the National Research Council under the Collaboration in Basic Science and Engineering Program. I would also like to thank D.Singleton for his invitation to research at VCU.

APPENDIX A: RIEMANN-CARTAN GEOMETRY

Here we give the necessary notions of Riemann-Cartan geometry as in Ref. [9]. The affine connection of Riemann-Cartan spacetime is:

$$\Gamma^{A}_{\bullet BC} = \left\{ {}^{B}_{AC} \right\} + S_{BC\bullet}^{\;\;\;\;A} - S_{C\bullet B}^{\;\;A} + S^{A}_{\bullet AB}, \tag{A1}$$

here $\left\{ {}^{B}_{AC} \right\}$ are Christoffel symbols. Cartan's torsion tensor $S_{BC\cdot}^{\;\;\;\;A}$ is defined according to :

$$S_{BC\bullet}^{\;\;\;\;A} = S^{A}_{\bullet BC} = \frac{1}{2}\Gamma^{A}_{[BC]} = \frac{1}{2}\left(\Gamma^{A}_{\bullet BC} - \Gamma^{A}_{\bullet CB}\right). \tag{A2}$$

The contorsion tensor is:

$$K_{BC\bullet}^{\;\;\;\;A} = S_{BC\bullet}^{\;\;\;\;A} + S_{C\bullet B}^{\;\;A} - S^{A}_{\bullet AB} \tag{A3}$$

In this case the affine connection is:

$$\Gamma^{A}_{\bullet BC} = \left\{ {}^{B}_{AC} \right\} - K_{BC\bullet}^{\;\;\;\;A}. \tag{A4}$$

The Riemann curvature tensor is defined in the usual way as:

$$\begin{aligned} R^{A}_{\bullet BCD} &= \partial_C \Gamma^{A}_{\bullet BD} - \partial_D \Gamma^{A}_{\bullet BC} + \Gamma^{A}_{\bullet EC}\Gamma^{E}_{\bullet BD} - \Gamma^{A}_{\bullet ED}\Gamma^{E}_{\bullet BC} = \\ \overset{\{\}}{R}{}^{A}{}_{\bullet BCD} &+ \nabla_D K^{A}_{\bullet BC} - \nabla_C K^{A}_{\bullet BD} + K^{A}_{\bullet EC} K^{E}_{\bullet BD} - K^{A}_{\bullet ED} K^{E}_{\bullet BC} \end{aligned} \tag{A5}$$

A modified torsion tensor is:

$$T_{BC\bullet}^{\;\;\;\;A} = S_{BC\bullet}^{\;\;\;\;A} + \delta^{A}_{B} S_{CD\bullet}^{\;\;\;\;D} - \delta^{A}_{C} S_{BD\bullet}^{\;\;\;\;D}. \tag{A6}$$

We can decompose the curvature scalar into Riemannian and contorsion pieces as follows:

$$R = \overset{\{\}}{R} + 2 \overset{\{\}}{\nabla}_A \left(K^{\bullet\;AB}_{B}\right) - T^{\bullet BC}_{A} K_{CB\bullet}^{\;\;\;\;A} \tag{A7}$$

For antisymmetric torsion we can write:

$$S_{ABC} = T_{ABC}; \qquad K_{ABC} = -S_{ABC}, \tag{A8}$$

$$R = \overset{\{\}}{R} - S_{ABC} S^{ABC}. \tag{A9}$$

APPENDIX B: CUBIC TERM IN 4D AND 5D HEISENBERG EQUATIONS

The calculation of $\left(\overline{\psi}\gamma^{[\alpha}\gamma^{\beta}\gamma^{\delta]}\psi\right)\left(\gamma_{[\alpha}\gamma_{\beta}\gamma_{\delta]}\psi\right)$ in 4D spacetime $(\alpha,\beta,\delta=0,1,2,3)$ give us the following result:

$$\left(\overline{\psi}\gamma^{[\alpha}\gamma^{\beta}\gamma^{\delta]}\psi\right)\left(\gamma_{[\alpha}\gamma_{\beta}\gamma_{\delta]}\psi\right)\propto e^{i\omega t}\left\{\begin{matrix} 6f(f^2-2g^2\sin^2\theta+g^2) \\ 12\sin\theta\cos\theta f g^2 e^{i\varphi} \\ -6i\cos\theta g(f^2+g^2) \\ 6i\sin\theta g(f^2-g^2)e^{i\varphi} \end{matrix}\right\}, \tag{B1}$$

$$\left(\overline{\psi}\gamma^{[\alpha}\gamma^{\beta}\gamma^{\delta]}\psi\right)\left(\overline{\psi}\gamma_{[\alpha}\gamma_{\beta}\gamma_{\delta]}\psi\right)\propto\left(f^4-4f^2g^2\sin^2\theta+2f^2g^2+g^4\right), \tag{B2}$$

and we see that term (B1) can not be included in Heisenberg equation (7) as it is inconsistent with spinor ansatz (10). In 5D spacetime the analogous cubic term $\left(\overline{\psi}\gamma^{[a}\gamma^{b}\gamma^{c]}\psi\right)\left(\gamma_{[a}\gamma_{b}\gamma_{c]}\psi\right)$ has the following form:

$$\left(\overline{\psi}\gamma^{[a}\gamma^{b}\gamma^{c]}\psi\right)\left(\gamma_{[a}\gamma_{b}\gamma_{c]}\psi\right)\propto 12e^{i\omega t}\left(f^2-g^2\right)\left\{\begin{matrix} f \\ 0 \\ ig\cos\theta \\ ig\sin\theta e^{i\varphi} \end{matrix}\right\}, \tag{B3}$$

$$\left(\overline{\psi}\gamma^{[a}\gamma^{b}\gamma^{c]}\psi\right)\left(\overline{\psi}\gamma_{[a}\gamma_{b}\gamma_{c]}\psi\right)\propto\left(f^2-g^2\right)^2 \tag{B4}$$

and the dependance on the polar angles θ and φ in (B3) is the same as in the ansatz (10).

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